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Scattering potentials with LS-terms from first-order Casimir operators

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Abstract. Using a *first-order* Casimir operator calculated in a non-standard realization for the $so(3,1)$ algebra, we obtain a one-dimensional scattering problem with LS-type interaction terms. It is shown that for this realization the square of this operator can be expressed in terms of the usual quadratic Casimir. Due to this constraint the scattering states are completely specified by restricting the possible set of eigenvalues accordingly. The results show that the use of extra Casimir operators can provide additional insight into the group theoretical structure of the scattering problem. A generalization for the $so(2n-1,1)$, $n>2$ case is also given. The underlying supersymmetry of the resulting Schrödinger equations is pointed out. The supersymmetric charge operators are related to our first-order Casimir operators.

1. Introduction

Since the seminal paper of Alhassid *et al* [1], the application of group theoretical methods for the description of quantum mechanical scattering problems has generated considerable interest. In these studies the quadratic Casimir of a non-compact group G with Lie algebra g was choosen to characterize the scattering process. Using different coordinate realizations for the generators of g, and the eigenvalue problem of the Casimir, one obtains Schrödinger equations with different interaction terms. Since G has unitary irreducible representations characterized by a continuous set of eigenvalues, it is possible to relate such values to the scattering energy.

For larger groups, however, we have also Casimir operators other then the quadratic Casimir. The number of independent Casimir operators equals the rank of g. For the realizations published in the literature so far these operators are either vanishing, or it is assumed that the scattering states are the ones annihilated by them [2,3]. The purpose of this paper is to point out that there are interesting realizations for which such operators are non-vanishing. By combining these additional Casimir operators with the quadratic Casimir, one can describe a more general set of scattering states, not necessarily annihilated by these extra operators.

The organization of this paper is as follows. In section 2. by using a special matrix valued realization for the rank two Lie algebra $so(3,1)$ we calculate both of the non-vanishing Casimir operators. One of them is the usual quadratic Casimir (C), the other is shown to be a first-order differential operator (C'). It turns out, however, that $(C')^2$ for the realization in question can be expressed in terms of C. This observation enables us to single out a restricted set of eigenvalues characterizing the scattering states of a one-dimensional scattering problem with LS type interaction terms. In section 3. we generalize

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our *first-order* Casimir operator (C') of section 2. for the Lie algebra $so(d,1)$, $d = 2n - 1$. Calculating the square of these operators we also obtain potentials of LS type with LS corresponding to the $SO(d)$ spin-orbit term. Such terms are precisely the ones obtained for the algebra $so(d,1)$ in [4] by calculating the *quadratic* Casimir (C). By demonstrating that (similar to the $so(3,1)$ case) $(C')^2$ and C are not independent, we clarify the connection between the results of [4] and this paper. We also point out the underlying supersymmetry of the resulting Schrödinger equations. The supersymmetric charge operators are related to our first-order Casimir operators. Some comments and conclusions are presented in section 4.

2. Casimir operators from a non-standard realization of $so(3,1)$

As a starting point we consider the following set of first-order matrix-valued differential operators:

$$J_{\alpha\beta} = L\delta_{\alpha\beta} + \tfrac{1}{2}\sigma_{\alpha\beta} \qquad M_{\alpha\beta} = K\delta_{\alpha\beta} + \frac{1}{1+X_4}\frac{1}{2}\sigma_{\alpha\beta} \times X \tag{2.1}$$

where

$$L_j = -\mathrm{i}\varepsilon_{jmn}X_m\partial_n \qquad K_j = -\mathrm{i}(X_4\partial_j + X_j\partial_4) \qquad j = 1,2,3 \tag{2.2}$$

and the coordinates $(X_j, X_4) \equiv (X, X_4)$, $j = 1,2,3$ lie on the upper-sheet of the hyperboloid H^3 defined by $-X^2 + (X_4)^2 = 1$, $\partial_j \equiv \partial/\partial X_j$, $\partial_4 \equiv \partial/\partial X_4$, σ_j are the usual 2×2 Pauli spin matrices. One can show that this set of generators satisfy the commutation relations of the $so(3,1)$ algebra

$$[J_j, J_m] = \mathrm{i}\varepsilon_{jmn}J_n \qquad [J_j, M_m] = \mathrm{i}\varepsilon_{jmn}M_n \qquad [M_j, M_m] = -\mathrm{i}\varepsilon_{jmn}J_n \tag{2.3}$$

where the matrix indices are left implicit. Such matrix valued realizations were introduced in [4,5] in order to describe a more general class of interaction terms group theoretically. From the mathematical point of view the operators of (2.1) are the generators of the induced representation for $so(3,1)$ induced by the spin-$\frac{1}{2}$ irrep of the $so(3) \sim su(2)$ subalgebra.

Since $SO(3,1)$ is a group of rank two, there are two independent Casimir operators. They are, the quadratic Casimir $C \equiv 2(J^2 - M^2)$, and $C' \equiv 8JM = 8MJ$, commuting with the six operators of (2.1). The origin of the numerical factors in the definition of C and C' will be clarified in section 3. Note that, had we used the operators L and K also spanning an $so(3,1)$ algebra, we would have obtained for $C' \equiv 0$. However, we will see that for the generators J and M $C' \neq 0$.

Let us now calculate C and C', using the coordinates $(\varrho, \theta, \varphi)$, $-\infty < \varrho < \infty$, $0 < \theta \leqslant \pi$, $0 < \varphi \leqslant 2\pi$ related to the ones (X, X_4) by

$$X = n(\theta,\varphi)\sinh\varrho \qquad X_4 = \cosh\varrho \qquad n^2 = 1 \tag{2.4}$$

where $n(\theta,\varphi) \equiv (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)$. We also introduce the operator

$$\mathcal{M} \equiv L \cdot \sigma + 1 \tag{2.5}$$

anticommuting with the operator $\sigma \cdot n \equiv \sigma_i n_i$, i.e. $\{\mathcal{M}, \sigma \cdot n\} = 0$. Moreover, one can prove that

$$\mathcal{M}(\mathcal{M} - 1) = L^2 \qquad \mathcal{M}(\mathcal{M} + 1) = 2J^2 - L^2 + \tfrac{1}{2}. \tag{2.6}$$

Using these results, straightforward calculation gives

$$C' \equiv TC'T^{-1} = -4\mathrm{i}(\sigma n)\left(\frac{\mathrm{d}}{\mathrm{d}\varrho} - \frac{\mathcal{M}}{\sinh\varrho}\right) \tag{2.7a}$$

$$\mathcal{C} \equiv TCT^{-1} = 2\left(\frac{\mathrm{d}^2}{\mathrm{d}\varrho^2} + \frac{\mathcal{M}(\mathcal{M}+1)}{4\cosh^2\frac{1}{2}\varrho} - \frac{\mathcal{M}(\mathcal{M}-1)}{4\sinh^2\frac{1}{2}\varrho} - \frac{3}{4}\right) \tag{2.7b}$$

where $T \equiv \sinh\varrho$ is a similarity transformation. This similarity transformation is convenient because the volume element for the hyperboloid is proportional to $\sinh^2\varrho$. So when using T this part of the volume element can be included into the wave function and one can use the volume element $\sin^2\theta \mathrm{d}\theta \mathrm{d}\varphi \mathrm{d}\varrho$. Note that according to (2.7*a*) C' is a *first-order* differential operator. Moreover, by virtue of the property $\{\mathcal{M}, \sigma n\} = 0$ it is also Hermitian, as it has to be.

The unitary irreducible representations of $SO(3,1)$ we are interested in are labelled by the pair of numbers (j_0, j_1) where $j_0 = 0, \pm\frac{1}{2}, \pm 1, \pm\frac{3}{2}, \ldots$ and $j_1 = ik$, $k \in \mathbb{R}$ [6]. Scattering states have to be chosen from the ones labelled by (j_0, j_1), and k is related to the scattering energy. Moreover denoting the possible set of scattering states by $|j_0, j_1\rangle$ we have

$$C|j_0, j_1\rangle = 2(j_0{}^2 + j_1{}^2 - 1)|j_0, j_1\rangle \tag{2.8a}$$

$$C'|j_0, j_1\rangle = -8\mathrm{i}j_0j_1|j_0, j_1\rangle. \tag{2.8b}$$

Due to our choice of coordinates we define $\langle\varrho, \theta, \varphi; \alpha|j_0, j_1\rangle \equiv u_{j_0,j_1;\alpha}(\varrho, \theta, \varphi)$ and the similarity transformed function $\Psi_{j_0,j_1;\alpha} = \sinh\varrho\, u_{j_0,j_1;\alpha}$. Note that $\Psi_{j_0,j_1;\alpha}$ is an eigenfunction of the operators $\mathcal{C}$ and $\mathcal{C}'$ with eigenvalues $2(j_0{}^2 + j_1{}^2 - 1)$ and $(-8\mathrm{i}j_0j_1)$, respectively.

Now we calculate the square of the operator $\mathcal{C}'$. Since $\{\mathcal{M}, \sigma n\} = 0$ and $(\sigma n)^2 = 1$ one can easily see that

$$\mathcal{C}'^2 = -16\left(\frac{\mathrm{d}^2}{\mathrm{d}\varrho^2} + \frac{\mathcal{M}(\mathcal{M}+1)}{4\cosh^2\frac{1}{2}\varrho} - \frac{\mathcal{M}(\mathcal{M}-1)}{4\sinh^2\frac{1}{2}\varrho}\right). \tag{2.9}$$

Comparing equations (2.7*b*) and (2.9) we get the relation

$$\tfrac{1}{4}\mathcal{C}'^2 + 2\mathcal{C} + 3 = 0 \tag{2.10}$$

implying by virtue of (2.8*a*), (2.8*b*) the constraint for the pair (j_0, j_1)

$$(j_0{}^2 - \tfrac{1}{4})(j_1{}^2 - \tfrac{1}{4}) = 0. \tag{2.11}$$

We are interested in irreps, hence we have to satisfy (2.11) from the restricted set for which $j_1 \equiv ik$. In this way we managed to single out the states labelled by the numbers $(\pm\frac{1}{2}, ik)$. It is easy to prove that these states are indeed scattering states. For this purpose we can use the eigenvalue problem of $\mathcal{C}'^2$ or $\mathcal{C}$, both of them resulting in the Schrödinger equation

$$\left(-\frac{\mathrm{d}^2}{\mathrm{d}\varrho^2} + \frac{\mathcal{M}(\mathcal{M}-1)}{4\sinh^2\frac{1}{2}\varrho} - \frac{\mathcal{M}(\mathcal{M}+1)}{4\cosh^2\frac{1}{2}\varrho}\right)\Psi_{\pm\frac{1}{2},ik,\lambda} = k^2\Psi_{\pm\frac{1}{2},ik,\lambda}. \tag{2.12}$$

In order to specify further this equation we have to solve the eigenvalue problem of the operator $\mathcal{M}$ acting merely on the angular part of $\Psi_{\pm\frac{1}{2},ik,\lambda}(\varrho,\theta,\varphi) \equiv R_{\pm\frac{1}{2},ik,\lambda}(\varrho)\Phi_\lambda(\theta,\varphi)$ as

$$\mathcal{M}\Phi_\lambda(\theta,\varphi) = \lambda\Phi_\lambda(\theta,\varphi). \tag{2.13}$$

According to (2.6) λ can be related to the eigenvalue of L^2 which is $l(l+1)$ hence for $l \geqslant 1$ there are *two* values of λ, i.e.

$$\lambda = \begin{cases} -l \\ l+1 \end{cases} \tag{2.14}$$

and for $l = 0$ the only possible value is $\lambda = 1$.

By virtue of (2.14) we obtain *two* Schrödinger equations

$$\left(-\frac{d^2}{d\varrho^2} + V_{l,\pm}(\varrho)\right) R_{\pm\frac{1}{2},ik,\lambda}(\varrho) = k^2 R_{\pm\frac{1}{2},ik,\lambda}(\varrho) \tag{2.15}$$

with potentials

$$V_{l,+}(\varrho) \equiv \frac{(l+\frac{1}{2})^2 - \frac{1}{4}}{4\sinh^2\frac{1}{2}\varrho} - \frac{(l+\frac{3}{2})^2 - \frac{1}{4}}{4\cosh^2\frac{1}{2}\varrho} \tag{2.16a}$$

$$V_{l,-}(\varrho) \equiv \frac{(l+\frac{1}{2})^2 - \frac{1}{4}}{4\sinh^2\frac{1}{2}\varrho} - \frac{(l-\frac{1}{2})^2 - \frac{1}{4}}{4\cosh^2\frac{1}{2}\varrho}. \tag{2.16b}$$

The second part of these potentials exhibits an implicit dependence on $j = l \pm \frac{1}{2}$. (For the relation between λ and j see equation (2.6).) After the change of variable $y = -\sinh^2\frac{1}{2}\varrho$ we can write $R_{\pm\frac{1}{2},ik,j}(\varrho)$ as

$$R_{\pm\frac{1}{2},ik,j}(y) = y^{\frac{l+1}{2}} \cdot \begin{cases} (1-y)^{\frac{l+2}{2}} F(ik, -ik; l+\frac{3}{2}; y) & \text{for } j = l+\frac{1}{2} \\ (1-y)^{\frac{l}{2}} F(1+ik, 1-ik; l+\frac{3}{2}; y) & \text{for } j = l-\frac{1}{2} \end{cases} \tag{2.17}$$

where $F(A,B;C;y)$ satisfies the hypergeometric differential equation. Exploiting the asymptotic properties [4] of F we can read off the reflection coefficient. The final result is

$$\mathcal{R}_j(k) = (-)^{n_j} \frac{\Gamma(j+1-ik)\Gamma(\frac{1}{2}+ik)}{\Gamma(j+1+ik)\Gamma(\frac{1}{2}-ik)} \tag{2.18}$$

where $n_j = 0$ when $j = l + \frac{1}{2}$, $n_j = 1$ when $j = l - \frac{1}{2}$.

3. Generalization for $SO(2n-1, 1)$

Having gained some knowledge from the $SO(3,1)$ case, now we try to write down a generalization of C' for the groups $SO(d,1)$, $d \equiv 2n-1$. First of all we need the $SO(d,1)$ generalization of the generators appearing in (2.1). According to the results of [5] the modified symmetry generators for the algebra $so(d,1)$ are

$$J_{kj} = L_{kj} + S_{kj} \qquad M_j = K_j + \frac{1}{1+X_{d+1}} X_k S_{kj} \tag{3.1}$$

where

$$L_{kj} = -\mathrm{i}(X_k\partial_j - X_j\partial_k) \qquad M_j = -\mathrm{i}(X_{d+1}\partial_j + X_j\partial_{d+1}) \qquad k, j = 1, \ldots, d. \tag{3.2}$$

Here $(X_j, X_{d+1}) \equiv (\boldsymbol{X}, X_{d+1})$ are coordinates on the upper sheet of the hyperboloid H^d defined by $-\boldsymbol{X}^2 + (X_{d+1})^2 = 1$, and $\partial_j \equiv \partial/\partial X_j$, $\partial_{d+1} \equiv \partial/\partial X_{d+1}$. The matrix-valued operators $(S_{jk})_{\alpha\beta}$ are chosen as

$$S_{ij} = \frac{1}{4\mathrm{i}}[\Gamma_i, \Gamma_j] \tag{3.3}$$

where the matrices $\Gamma_{j_{\alpha\beta}}$, $\alpha, \beta = 1, 2, \ldots, 2^{[d/2]} = 2^{n-1}$ are the generators of a d-dimensional Clifford algebra, i.e. we have

$$\{\Gamma_j, \Gamma_k\} = 2\delta_{jk}. \tag{3.4}$$

where the matrix indices $\alpha, \beta = 1, 2 \ldots, 2^{n-1}$ are implicit. The generators S_{ij} span an $so(d)$ algebra, i.e.

$$[S_{ij}, S_{kl}] = \mathrm{i}(\delta_{ik}S_{jl} + \delta_{jl}S_{ik} - \delta_{il}S_{jk} - \delta_{jk}S_{il}). \tag{3.5}$$

One can check that the generators J_{kj}, M_j satisfy the commutation relations of the $so(n, 1)$ algebra [5]

$$[J_{ij}, J_{kl}] = \mathrm{i}(\delta_{ik}J_{jl} + \delta_{jl}J_{ik} - \delta_{il}J_{jk} - \delta_{jk}J_{il}) \tag{3.6a}$$

$$[J_{ij}, M_k] = -\mathrm{i}(\delta_{jk}M_i - \delta_{ik}M_j) \tag{3.6b}$$

$$[M_j, M_k] = -\mathrm{i}J_{jk}. \tag{3.6c}$$

Note that the operator $\boldsymbol{MJ} = \boldsymbol{JM}$ of the previous section can be written as

$$C' = 8\boldsymbol{MJ} = \varepsilon_{\alpha\beta\gamma\tau}M^{\alpha\beta}M^{\gamma\tau} \qquad \alpha, \beta, \gamma, \tau = 1, 2, 3, 4 \tag{3.7}$$

where $M^{4i} \equiv M_i$ and $M^{ij} \equiv J_{ij} = \varepsilon_{ijk}J_k$. Hence a natural choice for C' for the groups $SO(2n-1, 1)$ is

$$C' = \varepsilon_{\alpha_1\beta_1\cdots\alpha_n\beta_n}M^{\alpha_1\beta_1}\ldots M^{\alpha_n\beta_n} \tag{3.8}$$

where $M^{d+1i} \equiv M_i$ and $M^{ij} \equiv J_{ij}$. In fact C' is a Casimir operator as can be easily verified by using the commutation relations. Moreover, it is a *first-order* differential operator having the form

$$C' = 2(-\mathrm{i})^{n-1}n!(2n-3)!!\left(M_i\Gamma_i - \frac{X_iL_{ij}\Gamma_j}{1+X_{2n}}\right)\Gamma_{2n} \tag{3.9}$$

where $\Gamma_{2n} = \Gamma_1\Gamma_2\cdots\Gamma_{2n-1}$. (For odd dimensions Γ_{d+1} is a number times the identity matrix.) The derivation of (3.9) is presented in the appendix. Note that had we used the usual operators L_{ij} and K_i instead of the matrix-valued ones J_{ij} and M_i, we would have obtained $C' = 0$. We also need the corresponding expression for the quadratic Casimir which is [5, 7]:

$$C = M_\alpha^\beta M_\beta^\alpha = J_{ij}J_{ij} - 2M_iM_i \tag{3.10}$$

where the indicies are raised and lowered by the metric $g_{\mu\nu} \equiv \mathrm{diag}(-1, -1, \ldots, -1, 1)$.

Now we define the d-dimensional generalization of the operator $\mathcal{M}$ as that satisfying $\{\mathcal{M}, \boldsymbol{\Gamma} \cdot \boldsymbol{n}\} = 0$:

$$\mathcal{M} \equiv L_{ij}S_{ij} + \left(\frac{d-1}{2}\right) \equiv \Sigma + \left(\frac{d-1}{2}\right). \tag{3.11}$$

We will need the expressions of the quantities $\frac{1}{2}L_{ij}L_{ij}$ and $\frac{1}{2}J_{ij}J_{ij}$ in terms of $\mathcal{M}$ later. In order to derive them we use the formulae [8]

$$\boldsymbol{L}^2 \equiv \tfrac{1}{2}L_{ij}L_{ij} = \Sigma(\Sigma - d + 2) \tag{3.12a}$$

and

$$\boldsymbol{J}^2 \equiv \frac{1}{2}J_{ij}J_{ij} = \Sigma(\Sigma - d + 1) + \frac{1}{4}\left(\frac{d(d-1)}{2}\right). \tag{3.12b}$$

Using these results we get

$$\boldsymbol{L}^2 = \mathcal{M}(\mathcal{M} - 1) - \tfrac{1}{4}(d-1)(d-3) \tag{3.13a}$$

$$\boldsymbol{J}^2 = \mathcal{M}^2 - \tfrac{1}{8}(d-1)(d-2). \tag{3.13b}$$

As a next step we calculate the operators C' and C using coordinates $(\varrho, \boldsymbol{n})$ $-\infty < \varrho < \infty$, related to the ones $(\boldsymbol{X}, X_{d+1})$ by

$$\boldsymbol{X} = \boldsymbol{n} \sinh\varrho \qquad X_{d+1} = \cosh\varrho \qquad \boldsymbol{n}^2 = 1 \tag{3.14}$$

as in (2.4), where $\boldsymbol{n}$ depends on $d-1$ polar coordinates of the unit sphere S^{d-1}. After straightforward calculation we get

$$C' \equiv TC'T^{-1} = 2(-\mathrm{i})^n n!(2n-3)!!(\boldsymbol{\Gamma}\boldsymbol{n})\left(\frac{\mathrm{d}}{\mathrm{d}\varrho} - \frac{\mathcal{M}}{\sinh\varrho}\right)\Gamma_{2n} \tag{3.15a}$$

$$C \equiv TCT^{-1} = 2\left(\frac{\mathrm{d}^2}{\mathrm{d}\varrho^2} + \frac{\mathcal{M}(\mathcal{M}+1)}{4\cosh^2\frac{1}{2}\varrho} - \frac{\mathcal{M}(\mathcal{M}-1)}{4\sinh^2\frac{1}{2}\varrho} - \frac{1}{8}d(d-1)\right) \tag{3.15b}$$

where we also performed the similarity transformation

$$T \equiv \left(\sinh\varrho\right)^{(d-1)/2} \tag{3.16}$$

for the same reason as in section 2.

In order to relate these operators to one-dimensional scattering problems we have to solve their eigenvalue problem. Moreover, we have to characterize the scattering states in terms of the unitary irreducible representations of $SO(2n-1, 1)$. These unireps have been classified in [6,9]. In these papers it was also argued that the same formulae for the eigenvalues of the Casimir operators can be used as in the case of of the corresponding compact groups $SO(2n)$. Using the results of these papers one can see that these irreps can be labelled by n-component weight vectors $\boldsymbol{m}$, and there are irreps for which

$m = (m_1, \ldots, m_{n-1}, \mathrm{i}k)$. After introducing the vector δ (half the sum of the positive roots) with components

$$\delta_i = n - i \qquad i = 1, \ldots, n \tag{3.17}$$

we can write the the following expressions for the eigenvalues of C and C' [7]:

$$C(m) = 2\sum_{i=1}^{n}(m_i m_i + 2m_i\delta_i) \tag{3.18a}$$

$$C'(m) = \mathrm{i}(-1)^{n(n-1)/2}2^n n!\prod_{i=1}^{n}(m_i + \delta_i)\,. \tag{3.18b}$$

We remark that for the groups $SO(2n-1,1)$ one can define $2n$ Casimir operators by the expressions

$$C_p = M^{\alpha_1}_{\alpha_2} M^{\alpha_2}_{\alpha_3} \cdots M^{\alpha_p}_{\alpha_1} \qquad p = 1, \ldots, 2n. \tag{3.19}$$

It can be shown that the operators for p odd are not independent hence we expect that the remaining n operators (their number is equal to the rank of $SO(2n-1,1)$) will completely characterize the irreps of $SO(2n-1,1)$. However, it can be shown that these operators cannot discriminate between all of the irreps hence we have to replace one of the operators by the operator C' of (3.8) [7]. Usually the operator to be replaced by C' is C_{2n}. Moreover we see that $C \equiv C_2$. Note that for the groups $SO(2n,1)$ no such substitution is needed.

Now we turn back to our original problem of interpreting the eigenvalue equations for C and C' as Schrödinger equations for one-dimensional scattering problems with LS terms. For this purpose we calculate the square of C' as in section 2. The result is

$$C'^2 = (-1)^n(2n!(2n-3)!!)^2\left(\frac{\mathrm{d}^2}{\mathrm{d}\varrho^2} + \frac{\mathcal{M}(\mathcal{M}+1)}{4\cosh^2\frac{1}{2}\varrho} - \frac{\mathcal{M}(\mathcal{M}-1)}{4\sinh^2\frac{1}{2}\varrho}\right)\Gamma_{2n}{}^2. \tag{3.20}$$

Comparing equations (3.18b) and (3.20) and using the result $\Gamma_{2n}{}^2 = (-1)^{n-1}$ we get the relation

$$(n!(2n-3)!!)^{-2}C'^2 + 2C + (2n-1)(n-1) = 0. \tag{3.21}$$

It is important to realize that equation (3.21) is merely one equation among the many possible others containing the remaining $n-2$ Casimir operators. Hence we cannot use equation (3.21) and the eigenvalue equations (3.18a), (3.18b) to obtain values for the components of the vector $m \equiv (m_1, \ldots, m_{n-1}, \mathrm{i}k)$ as in (2.11). (We have merely one equation for the $n-1$ unknowns.) However, by noting that

$$\frac{2^{n-1}}{(2n-3)!!} = \prod_{i=1}^{n-1}\frac{2}{2(n-i)-1}$$

by virtue of (3.18b) we can write

$$(2n!(2n-3)!!)^{-2}C'^2(m_1, \ldots, m_{n-1}, \mathrm{i}k) = k^2\left(\prod_{i=1}^{n-1}\frac{m_i + n - i}{-\frac{1}{2} + (n-i)}\right)^2. \tag{3.22}$$

Hence by choosing the weight vector m to be

$$m = (-\tfrac{1}{2}, -\tfrac{1}{2}, \ldots, -\tfrac{1}{2}, \mathrm{i}k) \tag{3.23}$$

the right-hand side of (3.22) reduces to k^2. Moreover, by calculating the eigenvalue $C(m)$ with the (3.23) choice using (3.18a) one can see that (3.21) is also satisfied. The final result for the eigenvalue problem of C'^2 reduces to

$$\left(-\frac{\mathrm{d}^2}{\mathrm{d}\varrho^2} + \frac{\mathcal{M}(\mathcal{M}-1)}{4\sinh^2\frac{1}{2}\varrho} - \frac{\mathcal{M}(\mathcal{M}+1)}{4\cosh^2\frac{1}{2}\varrho}\right)\Psi_{-\frac{1}{2},\ldots,-\frac{1}{2},\mathrm{i}k} = k^2\Psi_{-\frac{1}{2},\ldots,-\frac{1}{2},\mathrm{i}k}. \tag{3.24}$$

In order to write this equation as a pair of Schrödinger equations we have to solve the eigenvalue problem

$$\mathcal{M}\Phi_\lambda(n) = \lambda\Phi_\lambda(n). \tag{3.25}$$

of $\mathcal{M}$ as in section 2. Knowing the eigenvalue of L^2 which is $l(l+d-2)$ [7] and the (3.13a) expression relating L^2 and $\mathcal{M}$ we get for λ the values

$$\lambda = \begin{cases} -l - \dfrac{d-3}{2} \\ l + \dfrac{d-1}{2} \end{cases} \tag{3.26}$$

we can write down a pair of Schrödinger equations of the (2.15) form with scattering potentials

$$V_{l,d,+}(\varrho) \equiv \frac{(l+\frac{d}{2}-1)^2 - \frac{1}{4}}{4\sinh^2\frac{1}{2}\varrho} - \frac{(l+\frac{d}{2})^2 - \frac{1}{4}}{4\cosh^2\frac{1}{2}\varrho} \tag{3.27a}$$

$$V_{l,d,-}(\varrho) \equiv \frac{(l+\frac{d}{2}-1)^2 - \frac{1}{4}}{4\sinh^2\frac{1}{2}\varrho} - \frac{(l+\frac{d}{2}-2)^2 - \frac{1}{4}}{4\cosh^2\frac{1}{2}\varrho}. \tag{3.27b}$$

The calculation of the reflection coefficient is the same as in the previous section. The result is

$$\mathcal{R}_{j,d,\pm} = (-)^{d_j}\frac{\Gamma(\frac{1}{2}(d-1)+j-\mathrm{i}k)\Gamma(\frac{1}{2}+\mathrm{i}k)}{\Gamma(\frac{1}{2}(d-1)+j+\mathrm{i}k)\Gamma(\frac{1}{2}-\mathrm{i}k)} \tag{3.28}$$

where $n_j = 0$ when $j \equiv l + \frac{1}{2}$, $n_j = 1$ when $j \equiv l - \frac{1}{2}$. We see that for $d = 3$ we obtain the results of the previous section.

We close this section with the important observation that we can associate a supersymmetric partner Hamiltonian with the Hamiltonian of (3.24) in the following way. Let us define the operator A^- and its adjoint A^+ as the ones

$$A^\pm = \left(\pm\frac{\mathrm{d}}{\mathrm{d}\varrho} + W(\varrho)\right) \equiv \left(\pm\frac{\mathrm{d}}{\mathrm{d}\varrho} + \frac{\mathcal{M}}{\sinh\varrho}\right) \tag{3.29}$$

where $W(\varrho)$ is the superpotential. Note that these operators are matrix-valued first-order differential operators, since $\mathcal{M}$ is a $2^{n-1} \times 2^{n-1}$ matrix. As a next step we define the (supersymmetry) charge operator Q and its adjoint $Q^\dagger$ as the $2^{2n-2} \times 2^{2n-2}$ nilpotent matrices

$$Q = \begin{pmatrix} 0 & 0 \\ A^- & 0 \end{pmatrix} \qquad Q^\dagger = \begin{pmatrix} 0 & A^+ \\ 0 & 0 \end{pmatrix}. \tag{3.30}$$

By defining the supersymmetric Hamiltonian as

$$\mathcal{H} = \begin{pmatrix} H^+ & 0 \\ 0 & H^- \end{pmatrix} \equiv \begin{pmatrix} A^+A^- & 0 \\ 0 & A^-A^+ \end{pmatrix} \tag{3.31}$$

we can see that we are given a realization of the supersymmetry algebra

$$\mathcal{H} = \{Q, Q^\dagger\} \qquad Q^2 = 0 \qquad Q^{\dagger 2} = 0 \qquad [\mathcal{H}, Q] = [\mathcal{H}, Q^\dagger] = 0. \tag{3.32}$$

Note, however, that unlike the realizations usually used in the literature (see, e.g., [10, 11]) of supersymmetric quantum mechanics (SUSYQM), the operators $A^\pm$ are matrix valued. Hence our realization gives rise to a simple model of the multichannel supersymmetric scattering problem in the spirit of [12].

Moreover, it is interesting to see that our first-order Casimir operators turned out to be related to the supersymmetry generators Q and $Q^\dagger$. Indeed, in (3.30) we used $A^\dagger \sim \mathbf{\Gamma} \cdot \boldsymbol{n} C'$ (see equation (3.15*a*)). By calculating the Hamiltonians A^+A^- and A^-A^+ we obtain the interaction terms

$$V^\pm = \frac{\mathcal{M}(\mathcal{M} \pm 1)}{4\sinh^2 \frac{1}{2}\varrho} - \frac{\mathcal{M}(\mathcal{M} \mp 1)}{4\cosh^2 \frac{1}{2}\varrho} \tag{3.33}$$

that are supersymmetry partners of each other. One can see that H^- is just the Hamiltonian used in (3.24).

4. Conclusions

In this paper we initiated the examination of the possible role of Casimir operators other than the quadratic Casimir, in order to understand scattering problems characterized by a non-compact symmetry group G. As an example we considered the groups $SO(2n-1, 1)$. We used a matrix-valued realization for its Lie-algebra $so(2n-1, 1)$, which from the mathematical point of view is the representation induced by the spinor representation of the maximally compact subagebra $so(2n-1)$. For this realization it was shown that a non-vanishing Casimir operator (C') can be defined. Although this operator is of nth-order in the generators (see equation (3.8)) for the particular realization it turns out to be a *first-order* differential operator. This operator was used to single out a more general set of scattering states then the ones hitherto used in the literature. Moreover, these states are the ones corresponding to interaction terms of LS-type. Explicit expressions for the reflection coefficients were also given.

Note that our results are also valid for the groups $SO(2n, 1)$. Indeed, a straightforward calculation shows that the operator

$$\left(M_i\Gamma_i - \frac{X_i L_{ij}\Gamma_j}{(1 + X_{d+1})}\right)\Gamma_{d+1}$$

commutes with the operators of (3.1) (Γ_{d+1} for d even anticommutes with all Γ_i, hence it commutes with S_{kl} so with both J_{kl} and M_j), hence it is a Casimir operator irrespective of whether d is even or odd. However, the canonical form of this operator and its spectrum for d even (see equation (3.8) for d odd) is not known to the author.

We have also shown that our first-order Casimir operators can be used to obtain a matrix-valued realization of the SUSY algebra (3.32). According to the results of [10] this supersymmetry is not at all surprising, since our realization of the $so(2n-1)$ algebra is in terms of coordinates on the rank-one symmetric spaces $SO(2n-1,1)/SO(2n-1)$ which are the hyperboloids H^{2n-1}. On such spaces the quadratic Casimir C generates geometric motion, i.e. this operator can be regarded as the kinetic term when quantizing the classical geodesic motion [5], and shows explicit supersymmetry [10]. However, we have used a non-standard matrix-valued realization for C and we obtained a more general realization for the SUSY algebra than in [10]. It would be interesting to generalize these ideas of using vector-valued wavefunctions and the theory of induced representations in order to arrive at a group-theoretical description of scattering problems with coupled channels.

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Appendix. Calculation of the Casimir operator C'

Our starting point is the definition of C' as given by (3.8) of section 3. Recalling that $M^{d+1i} \equiv M_i$, and $M^{ij} \equiv J_{ij}$, $i, j = 1, \ldots, d = 2n-1$ we have

$$C' = 2n\varepsilon_{d+1 j_1 i_2 j_2 \cdots i_n j_n} M_{j_1} J_{i_2 j_2} \cdots J_{i_n j_n}. \tag{A1}$$

Since $J_{ij} = L_{ij} + S_{ij}$ and $\varepsilon_{ijk\cdots pq} M_i L_{jk} \cdots L_{pq} = 0$ due to the antisymmetry of $\varepsilon_{ijk\cdots pq}$, we get

$$C' = 2n\varepsilon_{ijklm\cdots pq} M_i (S_{jk} S_{lm} \cdots S_{pq} + (n-1) L_{jk} S_{lm} \cdots S_{pq}). \tag{A2}$$

Using the (3.3) and (3.1) definitions of S_{kl} and M_j and the (3.4) property of the Γ matrices we obtain

$$C' = \frac{(-1)^{n-1} n}{2^{n-2}} \left(\varepsilon_{ij\cdots q} M_i \Gamma_j \cdots \Gamma_q + \frac{(n-1)}{2(1+X_{d+1})} [X\Gamma, \varepsilon_{j\cdots spq} \Gamma_j \cdots \Gamma_s] L_{pq} \right). \tag{A3}$$

By virtue of the useful property of Γ matrices [13]

$$\Gamma_{i_1 i_2 \cdots i_k} = \kappa \varepsilon_{i_1 i_2 \cdots i_d} \Gamma^{i_{k+1} \cdots i_d} \Gamma_{d+1} \tag{A4}$$

$$\kappa = \frac{1}{(d-k)!} (-1)^{k(k-1)/2 + d(d-1)/2} \qquad \Gamma_{d+1} = \Gamma_1 \Gamma_2 \cdots \Gamma_d \tag{A5}$$

where $\Gamma_{i_1 i_2 \cdots i_k} \equiv \Gamma_{[i_1} \Gamma_{i_2} \cdots \Gamma_{i_n]}$ and the symbol $[i_1 \cdots i_n]$ denotes antisymmetrization in the corresponding indices, we have

$$\varepsilon_{ij\cdots q} \Gamma_j \cdots \Gamma_q = (d-1)! \Gamma_i \Gamma_{d+1} \tag{A6}$$

$$\varepsilon_{pqj\cdots s} \Gamma_j \cdots \Gamma_s = -\tfrac{1}{2}(d-2)! [\Gamma_p, \Gamma_q] \Gamma_{d+1} = -2\mathrm{i}(d-2)! S_{pq} \Gamma_{d+1}. \tag{A7}$$

(In the derivation of (A6), (A7) we have used the properties $\varepsilon_{ik_1\cdots k_{d-1}}\varepsilon_{jk_1\cdots k_{d-1}} = (d-1)!\delta_{ij}$ and $\varepsilon_{pqk_1\cdots k_{d-2}}\varepsilon_{rsk_1\cdots k_{d-2}} = (d-2)!(\delta_{pr}\delta_{qs} - \delta_{ps}\delta_{qr})$ of the Levi–Civita symbol in d dimensions.) Inserting these results in (A3) and calculating the commutator $[X\Gamma, S_{pq}]$ using

$$[S_{pq}, \Gamma_r] = -\mathrm{i}(\delta_{qr}\Gamma_p - \delta_{pr}\Gamma_q) \tag{A8}$$

we obtain $(d = 2n-1)$

$$C' = \frac{(-\mathrm{i})^{n-1}n}{2^{n-2}}(2n-2)!\left(M_i\Gamma_i - \frac{X_i L_{ij}\Gamma_j}{1+X_{d+1}}\right)\Gamma_{2n} \tag{A9}$$

which by virtue of the relation $2^{2-n}(2n-2)! = 2(n-1)!(2n-3)!!$ is expression (3.9) of section 3.

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